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Structural shakedown for elastic–plastic materials with hardening saturation surface

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Abstract

An elastic–plastic material model with internal variables and thermodynamic potential, not admitting hardening states out of a saturation surface, is assumed as a basis to formulate a statical Melan-type shakedown theorem. Grounding on the optimality conditions relative to the shakedown load multiplier problem for a structure subjected to cyclic loads, the impending inadaptation collapse mechanism at the shakedown limit state is analyzed and discussed. It is shown that the adopted model is able to catch ratchetting collapse mode at a structural level. Numerical results for a simple structure are finally reported.

1. Introduction

Shakedown analysis considers elastic–plastic solid bodies or structures subjected to variable repeated loads and studies the safety condition under which the structure's response is characterized by a bounded amount of the overall plastic dissipation work, namely, plastic strains are produced just for an initial loading stage followed by a complete purely elastic response to any further loading process. This condition, called shakedown, or adaptation, guarantees the safety of the structure against a kind of critical state referred to as non-shakedown or inadaptation, in which, even if the loads are always below the instantaneous plastic collapse value, the structure suffers continuous production of plastic strains.

The shakedown condition can be considered either with reference to a cyclically variable load or, more in general, to an arbitrarily variable load, but ranging within a fixed (convex) domain π. The latter loading scheme is only formally different from the former, as in fact, it has already been proved (Borkowsky and Kleiber, 1980; König, 1987; Polizzotto, 1982) that if shakedown occurs for a cyclic load path travelling on the load domain boundary, so called *Envelope Load History* (Polizzotto, 1982; Panzeca and Polizzotto, 1988), then it will occur for any load path (even not cyclic) inside the convex load domain. Hereinafter we refer to a cyclic load programme without losing generality.

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Experimental tests on simple steel structures have shown that, as the cyclic load exceeds some threshold, two kinds of critical states can be envisaged, Ponter (1983). The first is characterized by the fact that plastic strains increase cycle after cycle so that, after a certain number of cycles, the net accumulation of plastic strains somewhere in the structure will exceed the material ductility limit, or it becomes intolerably large for serviceability. This inadaptation mechanism is known as *incremental collapse mode* or *ratchetting*. The second critical state exhibits a continuous production of plastic strains along the cycle as the former but with zero plastic strain increment in the cycle. This inadaptation mechanism gives rise to low-cycle fatigue phenomena and it is known as *alternating plasticity collapse mode* or *plastic shakedown*.

The classical shakedown analysis based on perfectly-plastic material models is fully able to catch these two kinds of behaviour (Gokhfeld and Cherniavsky, 1980; Polizzotto, 1993). However for a better material modelling hardening has to be considered. The first attempt to introduce hardening into shakedown analysis goes back to Melan (refer to Koiter, 1960), who considered unlimited (i.e. indefinite capacity of) linear kinematic hardening. Neal (1950) proposed a one-dimensional overlay model based on the Masing rule able to account for limited kinematic hardening. Maier (1973) introduced unlimited linear hardening for piecewise linear elasto-plasticity with interacting yield planes. More recently, Maier (1987), Maier and Novati (1987) considered the case of non-linear isotropic hardening. The list of papers in this context could be richer, but the main point we would like to bring into prominence is that introduction of hardening material behaviour gives rise to some drawbacks, namely:

(i) for unlimited linear or non-linear isotropic hardening, shakedown will always occur, even if several loading cycles may be required and a large amount of plastic strains produced;
(ii) for unlimited linear or non-linear kinematic hardening the only kind of inadaptation mechanism that can be found is of alternating plasticity type.

Basically, every time unlimited hardening is allowed, the ratchetting collapse mode is ruled out against experimental evidence (König, 1987; Ponter, 1975).

In order to avoid such a disadvantage it is essential to bound the material hardening capacity in such a way that eventually perfectly-plastic behaviour is recovered after a certain straining process. Weichert and Gross-Weege (1988) proposed a shakedown theory based on internal variable plasticity models and two-surface plasticity theory; alternatively Stein et al. (1992) proposed an overlay model based on a generalization of the Masing rule and also presented an effective numerical procedure to face the statical shakedown problem. Polizzotto et al. (1991) presented a complete shakedown formulation for material models with internal variable, discussing also the shortcoming obtained for some constitutive models. A relevant contribution has been recently given by Pycko and Maier (1995), in the context of nonassociative hardening elastic–plastic material models, in which the concept of "asymptotic saturation state", giving rise to a convex bounding surface in the internal variable space, is employed.

In this paper the shakedown problem is formulated making use of a constitutive elastic–plastic material model, Polizzotto and Fuschi (1995), Fuschi and Polizzotto (1996), with internal variables and thermodynamic potential, not admitting hardening states out of a *saturation surface*. This surface, defined in the internal variables space (static internal variables space $\boldsymbol{\chi}$), bounds the hardening capacity of the material avoiding the drawbacks above-mentioned. The existence of such a saturation surface or, equivalently, the existence of a finite hardening domain in the static internal variables space is a direct consequence of the boundedness of the energy that can be stored in the material internal micro-structure. As shown in Polizzotto and Fuschi (1995), the definition of a saturation surface implies the existence of a bounding surface in the stress space as the envelope surface of all the yield surface configurations corresponding to hardening states not external to the hardening surface. With this assumption, the material behaves like a standard one as long as its hardening state either is not saturated, or undergoes a desaturation from a saturated hardening state; but it *behaves as a perfectly-plastic* material for a class of saturated hardening states for which the stress state lies on the bounding/limit surface.

The purpose of this paper is to discuss the relevant features of the shakedown analysis and of the steady-state response under cyclic loads for structures made of elastic–plastic material with saturation surface. The analyses hereafter presented do not substantially differ from the ones already proposed in the literature for different constitutive laws and are in fact performed particularizing standard procedures to the adopted constitutive model. Nevertheless such analyses are here performed with the main goal of showing that the adoption of a material model endowed with saturation surface allows one to reproduce ratchetting collapse mode at a structural level. Such a collapse mode is one of the relevant phenomena exhibited by hardening materials suffering cyclic load programs. To this aim, the classical static shakedown theorem (i.e. Melan's theorem) in the context of internal variable plasticity theory (Maier, 1987; Comi and Corigliano, 1991; Polizzotto et al., 1991) is reviewed. The problem of the shakedown load multiplier—evaluation of the maximum value of the load multiplier s, s_a say, such that shakedown occurs for a load domain amplified by $s \leqslant s_a$—is investigated, showing that some indeterminancies or unbounded cases, related to unlimited hardening material behaviour, are ruled out for the adopted formulation. Moreover, borrowing a procedure already implemented in Panzeca and Polizzotto (1988), in case of elastic perfectly-plastic solids subjected to cyclic loads, the impending inadaptation mechanism at shakedown limit state ($s = s_a$) is studied for elastic–plastic general isotropic/kinematic hardening solids subjected to cyclic loads. The impending collapse mechanism is, in fact, described by the Euler–Lagrange equations related to the problem of the shakedown load multiplier. A physical interpretation of the relevant conditions whether in the stress space or in the static internal variable space is given.

It is worth noting that when a shakedown analysis is performed for design purposes it is also important to make some assessment on the cumulated plastic strains up to a shakedown condition which eventually takes place. Such a check is needful to avoid unserviceability of the structure before shakedown occurs. This task can be accomplished either by a step-by-step analysis for an assigned load program, likely cumbersome and time-consuming, or by the use of an appropriate bounding technique which is often adequate, at least for very preliminary evaluations. These aspects are not pursued here because they are not the purpose of this paper.

The plan of the paper is as follows. In Section 2 the adopted material model is described. Section 3 is devoted to the static shakedown theorem while Section 4 addresses the problem of determining the shakedown load multiplier via statical approach, deriving the relevant equations. In Section 5 the nature of the impending inadaptation mechanism at the shakedown limit is discussed and interpreted whether in the stress space or in the static internal variable space. Finally, in Section 6 some numerical results relative to a simple structure are reported.

Notation—A compact notation is used. Boldface letters are used for vectors and tensors. The scalar product between vectors and tensors is denoted by dot (·) and colon (:) symbols, namely:

$\mathbf{u}\cdot\mathbf{v} = u_i v_i$, $\boldsymbol{\sigma}:\boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ji}$, $(\mathbf{C}:\boldsymbol{\varepsilon})_{ij} = C_{ijhk}\varepsilon_{kh}$, where the indices denote Cartesian components and the summation rule applies for repeated indices. The symbol (:=) means equality by definition, while a superimposed dot means derivative with respect to time t. Other symbols will be defined where they appear for the first time.

2. The material constitutive model

A material with mixed kinematic/isotropic hardening is considered. For simplicity, the effects of temperature changes on material data are disregarded. The related dual internal variables (Martin, 1975; Lemaitre and Chaboche, 1990; Lubliner, 1990) are denoted as $\boldsymbol{\chi} = (\mathbf{X}, Y)$ and $\boldsymbol{\xi} = (\boldsymbol{\alpha}, \beta)$ where $\mathbf{X}$ = back-stress tensor with the associated kinematic internal variable tensor $\boldsymbol{\alpha}$ and Y = isotropic stress scalar variable with related kinematic internal variable β. The material is characterized by a free energy as:

$$\Psi = \tfrac{1}{2}\boldsymbol{\varepsilon}^e:\mathbf{C}:\boldsymbol{\varepsilon}^e + \Psi_{\text{in}}(\boldsymbol{\xi}) \tag{1}$$

where $\boldsymbol{\varepsilon}^e$ is the elastic strain tensor that, making use of the decomposition of total strain $\boldsymbol{\varepsilon}$ into the reversible or elastic strain $\boldsymbol{\varepsilon}^e$ and the irreversible or plastic strain $\boldsymbol{\varepsilon}^p$, is expressed as $\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p$, $\mathbf{C}$ is the usual fourth order tensor of elastic moduli with its usual symmetries. The first term on the r.h.s. in eqn (1) is a convex positive-definite thermodynamic potential assumed as a quadratic function in the components of the strain tensor; it represents the elastic strain energy per unit volume. The thermodynamic potential $\Psi_{\text{in}}(\boldsymbol{\xi})$ is assumed to be convex and differentiable in the $\boldsymbol{\xi}$-space. In a plastic straining process, the value of $\Psi_{\text{in}}(\boldsymbol{\xi})$ at a time $t > 0$ represents the energy density required to promote the internal micro-structure mechanisms responsible for the change of the hardening state from the virgin one at $t = 0$ (where $\boldsymbol{\chi} = \boldsymbol{\xi} = \mathbf{0}$ and $\Psi_{\text{in}}(\boldsymbol{\xi}) = 0$) to that at time t.

The following laws of state hold:

$$\boldsymbol{\sigma} = \mathbf{C}:\boldsymbol{\varepsilon}^e, \quad \boldsymbol{\chi} = \frac{\partial \Psi_{\text{in}}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}. \tag{2a,b}$$

Equations (2a,b) express the duality between the static and the kinematic variables.

As observed in Polizzotto and Fuschi (1995), experimental uniaxial stress–strain curves for hardening materials show that the plastic stiffness decreases and tends to vanish as plastic strain monotonically increases, that is the internal mechanisms occur with decreasing energy storage rate so that the stored energy density $\Psi_{\text{in}}(\boldsymbol{\xi})$ tends to a finite constant value. Grounding on this phenomenological behaviour, valid for a certain class of ductile materials, it is possible to postulate the existence of a finite *hardening domain* in the $\boldsymbol{\xi}$-space defined by the inequality $\Psi_{\text{in}}(\boldsymbol{\xi}) \leqslant \bar{\Psi}$, with $\bar{\Psi}$ = material constant. Defining: $\bar{G}(\boldsymbol{\xi}) = \Psi_{\text{in}}(\boldsymbol{\xi}) - \bar{\Psi} \leqslant 0$, a saturation function in the kinematic internal variable space is obtained. The latter, since eqn (2b) holds, can be alternatively enforced in the static internal variable space as $G(\boldsymbol{\chi}) \leqslant 0$. The condition $G(\boldsymbol{\chi}) \leqslant 0$ introduces a hardening domain in the $\boldsymbol{\chi}$-space and an implicit limit upon the thermodynamic potential $\Psi(\boldsymbol{\xi})$. The hardening domain in the $\boldsymbol{\chi}$-space is surrounded by a surface $G(\boldsymbol{\chi}) = 0$ called *saturation surface* of the material, which collects all the hardening states $\boldsymbol{\chi}$ at the saturation limit. With these assumptions, hardening states out of the hardening surface are not allowed for the considered material. As long as $G(\boldsymbol{\chi}) < 0$, i.e. the hardening state is below the saturation limit, the plastic behaviour of the material can be described by standard internal-variable constitutive equations, Halphen and Nguyen (1975). When $G(\boldsymbol{\chi}) = 0$, i.e. the hardening state is at the saturation limit, the material plastic behaviour deviates from the standard one because—if no return occurs from the saturated hardening state—the hardening material behaviour is characterized by a static internal variable $\boldsymbol{\chi}$ moving on the saturation surface.

Let the yield and saturation functions be of the form:

$$f = f(\boldsymbol{\sigma}, \boldsymbol{\chi}) := f_0(\boldsymbol{\sigma} - \mathbf{X}) - f_1(Y) \tag{3}$$

$$G = G(\boldsymbol{\chi}) := G_0(\mathbf{X}) - G_1(Y) \tag{4}$$

respectively. Both functions are convex with respect to their arguments and the subscripts 0 and 1 refer to the kinematic and isotropic part, respectively. For a material obeying the normality rule (associative plasticity), the maximum intrinsic dissipation theorem (Martin, 1975; Lubliner, 1990) is the tool for deriving the pertinent flow laws, where, in this context, the plastically admissibility conditions are $f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0$, $G(\boldsymbol{\chi}) \leqslant 0$. Considering the internal variables altogether we get:

$$D(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}) = \max_{(\sigma,\chi)} (\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \boldsymbol{\chi} : \dot{\boldsymbol{\xi}}) \text{ subject to: } \quad f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0, \quad G(\boldsymbol{\chi}) \leqslant 0 \tag{5}$$

where $D(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}})$ is the intrinsic dissipation function related to a fixed plastic deformation mode specified by the plastic strain rate tensor $\dot{\boldsymbol{\varepsilon}}^p$ and the kinematic internal variable rate tensor $\dot{\boldsymbol{\xi}}$. The material flow laws are provided by the Kuhn–Tucker conditions of problem (5), i.e.

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \dot{\boldsymbol{\xi}} = -\dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\chi}} - \dot{\mu} \frac{\partial G}{\partial \boldsymbol{\chi}} \tag{6a,b}$$

$$f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda} f(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0 \tag{7a–c}$$

$$G(\boldsymbol{\chi}) \leqslant 0, \quad \dot{\mu} \geqslant 0, \quad \dot{\mu} G(\boldsymbol{\chi}) = 0 \tag{8a–c}$$

with $\dot{\lambda}$, $\dot{\mu}$ plastic and hardening coefficients, respectively. By inspection of eqns (6) it arises: (i) the plastic strain rate, $\dot{\boldsymbol{\varepsilon}}^p$, is, as for a standard material, normal to the yield surface; (ii) the kinematic internal variable rate, $\dot{\boldsymbol{\xi}}$, is the sum of two contributions, one of which is normal to the yield surface, the other is normal to the saturation surface $G(\boldsymbol{\chi}) = 0$. That is, with the present material model, $\dot{\boldsymbol{\xi}}$ can vanish even if $\dot{\boldsymbol{\varepsilon}}^p \neq 0 (\dot{\lambda} > 0)$ but for a hardening state at the saturation limit ($\dot{\mu} > 0$); it is not so for a standard material (not endowed with saturation surface) in which $\dot{\boldsymbol{\xi}} = \mathbf{0}$ if and only if $\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$. Finally, eqns (7) and (8) are the plastic and the saturation loading/unloading conditions, respectively, which $\dot{\lambda}$ and $\dot{\mu}$ have to comply with.

As a consequence of the convexity of $f(\boldsymbol{\sigma}, \boldsymbol{\chi})$ and $G(\boldsymbol{\chi})$ the following inequality also holds:

$$(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}) : \dot{\boldsymbol{\varepsilon}}^p - (\boldsymbol{\chi} - \bar{\boldsymbol{\chi}}) : \dot{\boldsymbol{\xi}} \geqslant 0. \tag{9}$$

The latter expresses, in the present context, an extension of the Drucker stability postulate where $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ and $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}})$ correspond to each other through eqns (6)–(8) while $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$ are any set of plastically admissible stresses and static internal variables, i.e. $f(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}) \leqslant 0$ and $G(\bar{\boldsymbol{\chi}}) \leqslant 0$. Condition (9) holds as an equality either for $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}$ and $\bar{\boldsymbol{\chi}} = \boldsymbol{\chi}$, in which case $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\boldsymbol{\xi}}$ may be nonvanishing, or for $\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$ and $\dot{\boldsymbol{\xi}} = \mathbf{0}$, in which case $\boldsymbol{\sigma}$ and $\boldsymbol{\chi}$ may differ from $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\chi}}$, respectively, provided f and G are smooth (as implicitly assumed).

For a complete discussion of the material model and a detailed characterization of eqns (6)–(8)

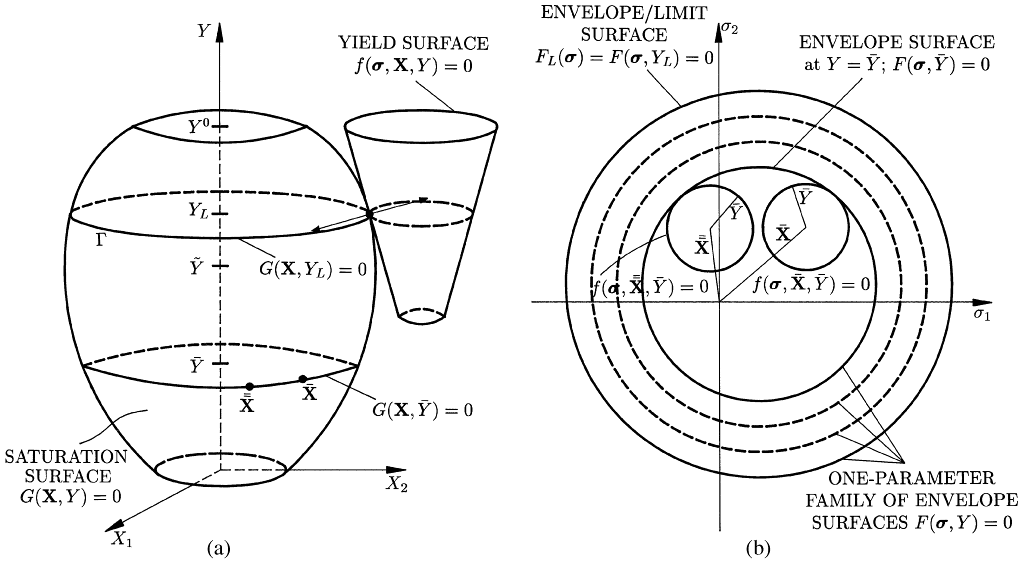

Fig. 1. Geometrical sketch representing: (a) the saturation surface and the yield surface in the $\boldsymbol{\chi}$-space; and (b) the yield, envelope and limit surfaces in the $\boldsymbol{\sigma}$-space.

refer to Fuschi and Polizzotto (1996). For subsequent use attention is now focused on a geometrical description of the adopted material model whether in the $\boldsymbol{\chi}$-space or in the $\boldsymbol{\sigma}$-space. To this aim we assume a convex saturation surface as the one depicted in Fig. 1(a). With reference to the figure we can assert the following. As long as the point $(\mathbf{X}, Y)$ belongs to the saturation surface, $G(\mathbf{X}, Y) = 0$, while taking $Y = \bar{Y} = \text{const}$ [e.g. points like $(\bar{\mathbf{X}}, \bar{Y})$ and $(\bar{\bar{\mathbf{X}}}, \bar{Y})$ in Fig. 1(a)], the yield surface $f = 0$ correspondingly moves in the $\boldsymbol{\sigma}$-space [e.g. $f(\boldsymbol{\sigma}, \bar{\mathbf{X}}, \bar{Y}) = 0$ and $f(\boldsymbol{\sigma}, \bar{\bar{\mathbf{X}}}, \bar{Y}) = 0$, respectively, in Fig. 1(b)] giving rise to an envelope surface $F(\boldsymbol{\sigma}, \bar{Y}) = 0$, whose position and shape depends on $\bar{Y}$ (Fig. 1(b)). On letting Y to vary in such a way that point $(\mathbf{X}, Y)$ moves wherever on $G = 0$, a one-parameter family of envelope surfaces $F(\boldsymbol{\sigma}, Y) = 0$ is generated in $\boldsymbol{\sigma}$-space. Moreover it is possible to demonstrate that this family is enveloped by a *limit surface*—envelope surface of *all* the yield surface configurations corresponding to admissible hardening states—whose expression is $F_L(\boldsymbol{\sigma}) = F(\boldsymbol{\sigma}, Y_L) = 0$ (Fig. 1(b)). The latter corresponds to points $(\mathbf{X}, Y)$ travelling on the curve $G(\mathbf{X}, Y_L) = 0$, curve Γ in Fig. 1(a), which collects the set of so-called *critical hardening states*. The peculiarity of these points is the following. If we assume a material which exhibits positive-isotropic hardening—i.e. softening is excluded—whose yield function is expressed by eqn (3) in which a linear dependence on Y is also assumed, the yield function $f(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$ possesses in the $\boldsymbol{\chi}$-space a conical shape representation (Fig. 1(a)). The curve Γ is the points' locus at which the yield function is in full external contact with the saturation function $G(\boldsymbol{\chi}) = 0$, i.e. it collects points $(\mathbf{X}, Y_L)$ at which the yield and the saturation functions admit the same normal. Of course the position and shape of curve Γ is related to the assumed forms of the yield and the saturation functions, $f(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$ and $G(\boldsymbol{\chi}) = 0$, respectively, and it must be consistently located. In what follows, for simplicity, we exclude softening material behaviour and assume $f_1(Y)$ in eqn (3) as a linear function of Y.

A geometrical sketch of the mutual relationships existing between the saturation surface and the envelope surface at an arbitrarily chosen $Y = \bar{Y}$ is drawn in Figs 2 and 3, respectively. A non-saturated state $(\bar{\mathbf{X}}, \bar{Y})$ is considered [i.e. $G(\bar{\mathbf{X}}, \bar{Y}) < 0$] and two stress states, $\bar{\boldsymbol{\sigma}}$ and $\bar{\bar{\boldsymbol{\sigma}}}$, at the yield limit are represented in Fig. 2(a). Both of them allow plastic strain rates which develop following the normality rule [eqn (6a)]. Correspondingly in Fig. 2(b) two different positions of the yield surface in $\mathbf{X}$-space are located for $\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma} = \bar{\bar{\boldsymbol{\sigma}}}$, respectively. Both surfaces contain the static internal variable state $\bar{\mathbf{X}}$, and the inner normals to them evaluated at $\bar{\mathbf{X}}$ give the only-plastic-contribution $(\dot{\lambda}\, \partial f/\partial \mathbf{X})$ to the kinematic internal variable rate $\dot{\boldsymbol{\alpha}}$. A saturated state is considered (i.e. $G(\bar{\mathbf{X}}, \bar{Y}) = 0$) and, as before, two stress states, $\bar{\boldsymbol{\sigma}}$ and $\bar{\bar{\boldsymbol{\sigma}}}$, at the yield limit are represented in Fig. 3(a), with $\bar{\boldsymbol{\sigma}}$ lying also on the relevant envelope surface $F(\boldsymbol{\sigma}, \bar{Y})$. Correspondingly in the $\mathbf{X}$-space, Fig. 3(b), two positions of the yield surface can be located as above. However, looking at the kinematic internal variable rate $\dot{\boldsymbol{\alpha}}$, besides the plastic contribution in a nonvanishing saturation-nature contribution, along the inner normal to $G(\mathbf{X}, \bar{Y})$ at $\bar{\mathbf{X}}$, arises. In this case the contributions to $\dot{\beta}$ are given in Fig. 4(a, b) for $\bar{Y} > Y_L$ and $\bar{Y} = Y_L$.

3. The shakedown problem

Let a continuous solid body, occupying the domain V surrounded by the surface S, be referred to a Cartesian orthogonal co-ordinate system $\mathbf{x} = (x_1, x_2, x_3)$ in its undisturbed initial state, and let the relevant constituent material obey eqns (2) and (6)–(8). The body is constrained on $S_u \subset S$ and is subjected to external actions as: body forces $\bar{\mathbf{b}}$ in V, imposed strains (e.g. thermal strains) $\boldsymbol{\varepsilon}^\theta$ in

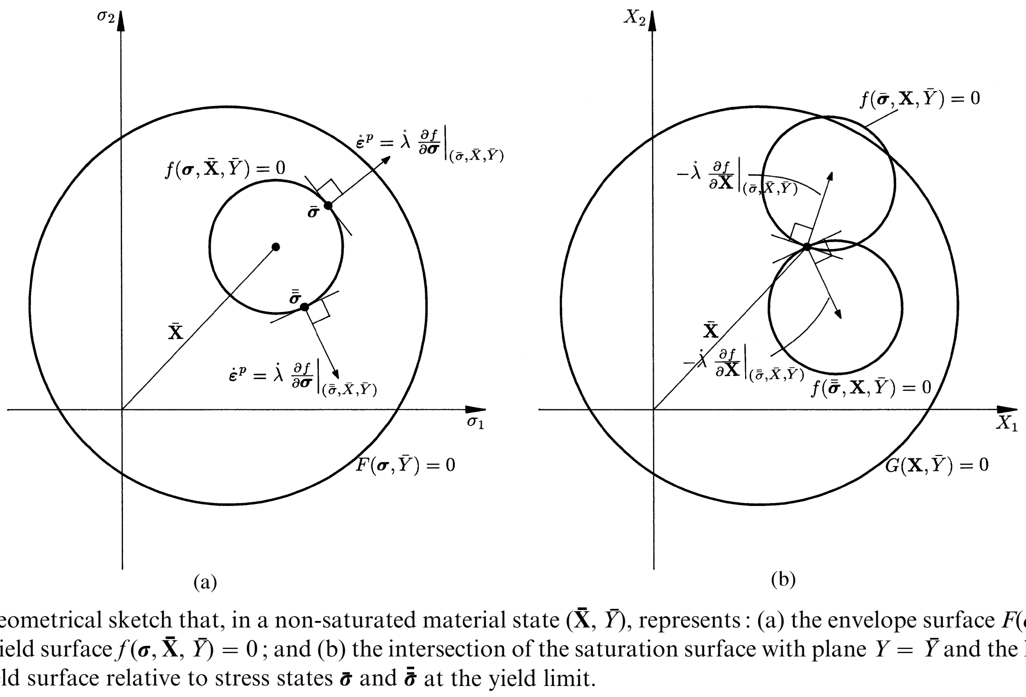

Fig. 2. Geometrical sketch that, in a non-saturated material state $(\bar{\mathbf{X}}, \bar{Y})$, represents: (a) the envelope surface $F(\boldsymbol{\sigma}, \bar{Y}) = 0$ and the yield surface $f(\boldsymbol{\sigma}, \bar{\mathbf{X}}, \bar{Y}) = 0$; and (b) the intersection of the saturation surface with plane $Y = \bar{Y}$ and the locations of the yield surface relative to stress states $\bar{\boldsymbol{\sigma}}$ and $\bar{\bar{\boldsymbol{\sigma}}}$ at the yield limit.

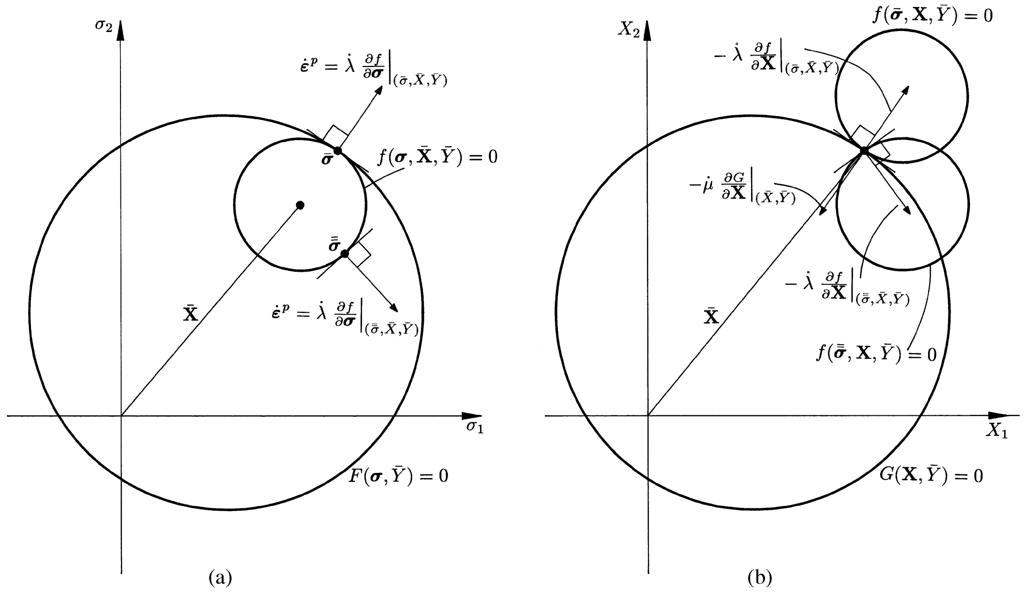

Fig. 3. Geometrical sketch that, in a saturated material state $(\bar{\mathbf{X}}, \bar{Y})$, represents: (a) the envelope surface $F(\boldsymbol{\sigma}, \bar{Y}) = 0$ and the yield surface $f(\boldsymbol{\sigma}, \bar{\mathbf{X}}, \bar{Y}) = 0$; and (b) the intersection of the saturation surface with plane $Y = \bar{Y}$ and the locations of the yield surface relative to stress states $\bar{\boldsymbol{\sigma}}$ and $\bar{\bar{\boldsymbol{\sigma}}}$ at the yield limit.

V, tractions $\bar{\mathbf{t}}$ on $S_t = S - S_u$, and imposed displacements $\bar{\mathbf{u}}$ on S_u. These external actions are linearly expressed in terms of a vector $\mathbf{P}$ of independent load parameters, and vary periodically with time period Δt, i.e. $\mathbf{P}(t+\Delta t) = \mathbf{P}(t)$ for all $t \geqslant 0$. For simplicity, $\mathbf{P}$ is viewed as two-dimensional in the

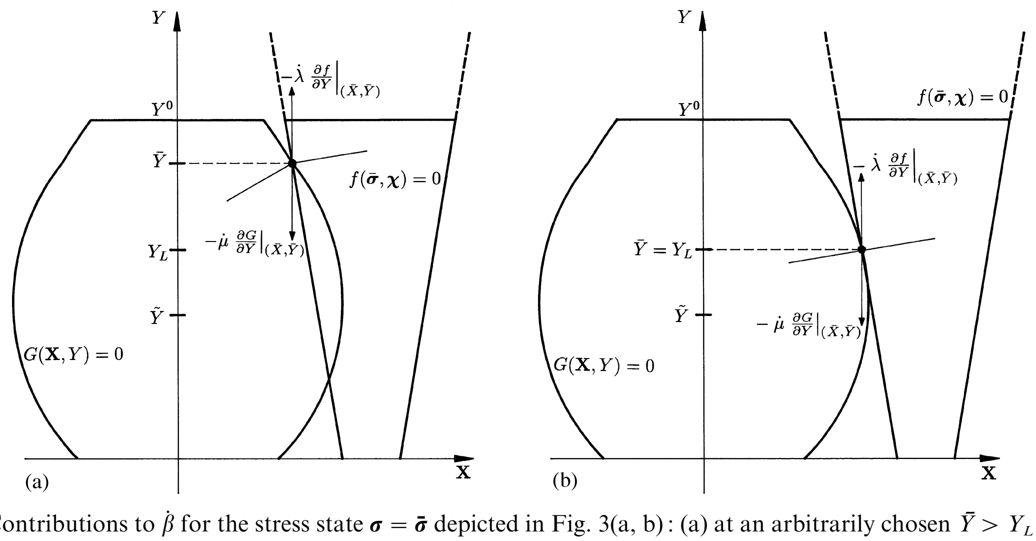

Fig. 4. Contributions to $\dot{\beta}$ for the stress state $\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}}$ depicted in Fig. 3(a, b): (a) at an arbitrarily chosen $\bar{Y} > Y_L$; and (b) at $\bar{Y} = Y_L$.

following, but this restriction can be easily removed. As already asserted in Section 1, the shakedown problem is herein discussed with reference to an assigned cyclically varying load; the latter, if that is the case, can be viewed as the *Envelope Load History* of a fixed convex load domain π. Moreover it is assumed that both geometric and inertia effects are negligible, such that the infinitesimal displacement theory for quasi-static loads is applicable. Thermal effects over the constitutive laws are also assumed negligible.

Under the above hypotheses, the shakedown problem consists in ascertaining those a priori conditions, if any, such that the structure subjected to the cyclic load $\mathbf{P}(t)$ will eventually behave elastically and that the overall plastic dissipation work produced in the transient (i.e. initial elastic–plastic) phase is bounded even if $t \to \infty$.

In the shakedown theory the relevant criteria for shakedown are the *statical theorem* of Melan and the *kinematical theorem* of Koiter (Koiter, 1960), both of which have already been generalized to non-linear hardening by Maier (1987), Maier and Novati (1987), and in the case of material models endowed with dual internal variables and thermodynamic potential by Comi and Corigliano (1991), Polizzotto et al. (1991). In the following, by standard arguments (Maier, 1987; Polizzotto et al., 1991; Corradi, 1994), the statical shakedown theorem is extended to the considered material model.

3.1. *Statical shakedown theorem*

A necessary and sufficient condition for shakedown to occur is that there exist a set $(\hat{\boldsymbol{\sigma}}^R, \hat{\boldsymbol{\chi}})$ of static variables, namely a residual stress field $\hat{\boldsymbol{\sigma}}^R(\mathbf{x})$ and a static internal variable field $\hat{\boldsymbol{\chi}}(\mathbf{x})$, both time-independent, such that

$$f(\boldsymbol{\sigma}^E(\mathbf{x}, \mathbf{P}(\tau)) + \hat{\boldsymbol{\sigma}}^R(\mathbf{x}), \hat{\boldsymbol{\chi}}(\mathbf{x})) \leqslant 0; \quad G(\hat{\boldsymbol{\chi}}(\mathbf{x})) \leqslant 0 \quad \text{in } V \times [0, \Delta T] \tag{10}$$

where $\boldsymbol{\sigma}^E(\mathbf{x}, \mathbf{P}(\tau))$ is the elastic response of the body to the load $\mathbf{P}(\tau)$ and τ denotes the time variable in the cycle, $0 \leqslant \tau \leqslant \Delta t$.

Necessity—It must be proven that, if shakedown occurs, then (10) hold. The necessity is self-evident as in fact at shakedown no further plastic strains and kinematic internal variables are produced in addition to those cumulated in the transient phase and the subsequent structural behaviour is elastic. On denoting as $\boldsymbol{\sigma}^*(\mathbf{x})$ and $\boldsymbol{\chi}^*(\mathbf{x})$ the residual stresses and the static internal variables, respectively, correspondent to the plastic strains and kinematic internal variables at the end $t^* \geqslant 0$ of the transient phase, the actual stresses $\boldsymbol{\sigma}(\mathbf{x}, t) = \boldsymbol{\sigma}^E(\mathbf{x}, \mathbf{P}(\tau)) + \boldsymbol{\sigma}^*(\mathbf{x})$ and static internal variables $\boldsymbol{\chi}(\mathbf{x}, t) = \boldsymbol{\chi}^*(\mathbf{x})$ evaluated at the (general) time $t = t^* + \tau$ comply with the constitutive equations and thus, eqns (10) are satisfied if one chooses $\hat{\boldsymbol{\sigma}}^R \equiv \boldsymbol{\sigma}^*$ and $\hat{\boldsymbol{\chi}} \equiv \boldsymbol{\chi}^*$. Q.E.D.

Sufficiency—It must be proven that under validity of (10) for some $\hat{\boldsymbol{\sigma}}^R(\mathbf{x})$ and $\hat{\boldsymbol{\chi}}(\mathbf{x})$ shakedown occurs. Let by hypothesis a set $(\hat{\boldsymbol{\sigma}}^R, \hat{\boldsymbol{\chi}})$ there exists such that conditions (10) hold. Let the actual response be denoted by symbols as $\boldsymbol{\sigma}(\mathbf{x}, t)$, $\dot{\boldsymbol{\varepsilon}}^p(\mathbf{x}, t), \ldots$ where $t \geqslant 0$ is the general time variable and let the stress argument of the first eqn (10) be denoted as follows:

$$\hat{\boldsymbol{\sigma}}(\mathbf{x}, \mathbf{P}(t)) := \boldsymbol{\sigma}^E(\mathbf{x}, \mathbf{P}(t)) + \hat{\boldsymbol{\sigma}}^R(\mathbf{x}). \tag{11}$$

Applying (9), valid in V and for $t \geqslant 0$ we can write:

$$J := (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \dot{\boldsymbol{\varepsilon}}^p - (\boldsymbol{\chi} - \hat{\boldsymbol{\chi}}) : \dot{\boldsymbol{\xi}} \tag{12a}$$

$$J(\mathbf{x}, t) \geqslant 0 \quad \text{in } V \quad \text{for all } t \geqslant 0. \tag{12b}$$

Making use of the identity $\dot{\boldsymbol{\varepsilon}}^p = \dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^E - \mathbf{A}:(\dot{\boldsymbol{\sigma}} - \dot{\hat{\boldsymbol{\sigma}}})$, where $\boldsymbol{\varepsilon}^E$ is the total strain in the elastic response and $\mathbf{A} = \mathbf{C}^{-1}$, and integrating over V, eqn (12a) can be posed in the form:

$$\int_V J\,\mathrm{d}V = \int_V (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}):(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^E)\,\mathrm{d}V - \int_V (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}):\mathbf{A}:(\dot{\boldsymbol{\sigma}} - \dot{\hat{\boldsymbol{\sigma}}})\,\mathrm{d}V - \int_V (\boldsymbol{\chi} - \hat{\boldsymbol{\chi}}):\dot{\boldsymbol{\xi}}\,\mathrm{d}V \tag{13}$$

Since the first integral on the r.h.s. vanishes by the virtual work principle applied to the self-equilibrated field $(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}})$ and the self-compatible field $(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^E)$ (i.e. compatible with zero displacements on S_u), eqn (13) can be rewritten as:

$$\int_V (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}):\mathbf{A}:(\dot{\boldsymbol{\sigma}} - \dot{\hat{\boldsymbol{\sigma}}})\,\mathrm{d}V + \int_V (\boldsymbol{\chi} - \hat{\boldsymbol{\chi}}):\dot{\boldsymbol{\xi}}\,\mathrm{d}V = -\int_V J\,\mathrm{d}V \tag{14}$$

Following a procedure first devised by Maier (1987), it can be easily shown that the two integrals on the l.h.s. of eqn (14) are equivalent to

$$\int_V (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}):\mathbf{A}:(\dot{\boldsymbol{\sigma}} - \dot{\hat{\boldsymbol{\sigma}}})\,\mathrm{d}V = \dot{L}^e, \qquad \int_V (\boldsymbol{\chi} - \hat{\boldsymbol{\chi}}):\dot{\boldsymbol{\xi}}\,\mathrm{d}V = \dot{L}^p \tag{15a,b}$$

where $L^e(t)$ and $L^p(t)$ have the following alternative expressions:

$$\begin{aligned} L^e(t) &= \frac{1}{2}\int_V (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}):\mathbf{A}:(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}})\,\mathrm{d}V|_t \\ &= \frac{1}{2}\int_V (\boldsymbol{\sigma}^R - \hat{\boldsymbol{\sigma}}^R):\mathbf{A}:(\boldsymbol{\sigma}^R - \hat{\boldsymbol{\sigma}}^R)\,\mathrm{d}V|_t \end{aligned} \tag{16a}$$

$$\begin{aligned} L^p(t) &= \int_V [\Psi(\boldsymbol{\xi}) - \Psi(\hat{\boldsymbol{\xi}}) - \hat{\boldsymbol{\chi}}:(\boldsymbol{\xi} - \hat{\boldsymbol{\xi}})]\,\mathrm{d}V|_t \\ &= \int_V [\Omega(\hat{\boldsymbol{\chi}}) - \Omega(\boldsymbol{\chi}) - \boldsymbol{\xi}:(\hat{\boldsymbol{\chi}} - \boldsymbol{\chi})]\,dV|_t \end{aligned} \tag{16b}$$

where $\boldsymbol{\sigma}^R = \boldsymbol{\sigma} - \boldsymbol{\sigma}^E$ denotes the actual residual stress field, and $\Omega(\boldsymbol{\chi}) = \boldsymbol{\chi}:\boldsymbol{\xi} - \Psi(\boldsymbol{\xi})$ is the Legendre transform of the thermodynamic potential $\Psi(\boldsymbol{\xi})$. Moreover L^e and L^p, hence $L := L^e + L^p$, are positive definite as a consequence of the positive definiteness of $\mathbf{A}$ and of the convexity of Ψ and Ω (Maier, 1987; Polizzotto et al., 1991). Therefore, eqn (14) becomes, in virtue of eqn (12b):

$$\dot{L}(t) = \dot{L}^e(t) + \dot{L}^p(t) = -\int_V J\,\mathrm{d}V \leqslant 0 \quad \text{for all } t \geqslant 0. \tag{17}$$

This result implies that the positive definite functional $L(t)$ decreases monotonically with time as long as $J > 0$, even in a small portion of V. Because L cannot assume negative values, there will certainly be a time $t = t^*$ (adaptation time) after which L assumes a constant value, i.e. $\dot{L} = 0$ for $t \geqslant t^*$, thus, by eqns (17) and (12b), it will be $J(\mathbf{x}, t) = 0$ identically after t^*, which is possible if, and only if, $\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$, $\dot{\boldsymbol{\xi}} = \mathbf{0}$ in V for $t \geqslant t^*$, that is, if and only if shakedown occurs. Q.E.D.

4. The shakedown load multiplier

Let the loads be specified to within a scalar factor $s \geqslant 0$, i.e. $\mathbf{P} = \mathbf{P}(t) = s\bar{\mathbf{P}}(t)$, where $\bar{\mathbf{P}}(t)$ is a reference load. The maximum value of s, s_a say, such that for any $s \leqslant s_a$ shakedown occurs, is the *shakedown load multiplier* of the given structure. The latter can in general be found using either Melan's or Koiter's theorems, but is here evaluated by application of Melan's theorem given in Section 3.

To this aim $\bar{\mathbf{P}}(t)$ is considered specified for $0 \leqslant t \leqslant \Delta t$ and thus, we set the problem:

$$s_a = \max_{(s,\hat{\sigma}^R,\hat{\chi})} s \quad \text{subject to:} \tag{18a}$$

$$f(\hat{\boldsymbol{\sigma}}(\mathbf{x}, t), \hat{\boldsymbol{\chi}}(\mathbf{x})) \leqslant 0 \quad \text{in } V \times [0, \Delta T] \tag{18b}$$

$$G(\hat{\boldsymbol{\chi}}(\mathbf{x})) \leqslant 0 \quad \text{in } V \tag{18c}$$

$$\hat{\boldsymbol{\sigma}}(\mathbf{x}, t) = s\boldsymbol{\sigma}^E(\mathbf{x}, \bar{\mathbf{P}}(t)) + \hat{\boldsymbol{\sigma}}^R(\mathbf{x}) \quad \text{in } V \times [0, \Delta T] \tag{18d}$$

$$\operatorname{div} \hat{\boldsymbol{\sigma}}^R(\mathbf{x}) = \mathbf{0} \quad \text{in } V, \quad \hat{\boldsymbol{\sigma}}^R(\mathbf{x}) \cdot \mathbf{n} = \mathbf{0} \quad \text{on } S_t \tag{18e,f}$$

where the cupped symbols $(\hat{\cdot})$ refer to the unknown static variables required by Melan theorem. More precisely, eqns (18b,c) are the admissibility conditions to be satisfied by the stresses $\hat{\boldsymbol{\sigma}}$ given by (18d) and by the static internal variables $\hat{\boldsymbol{\chi}}(\mathbf{x})$, respectively; eqns (18e,f) state that $\hat{\boldsymbol{\sigma}}^R$ are residual stresses; $\mathbf{n}$ is the outward unit normal to S.

Applying the Lagrange multiplier method, we consider the following augmented functional:

$$\Upsilon = -\kappa s + \int_0^{\Delta T} \int_V \dot{\lambda}(\mathbf{x}, t) f(\hat{\boldsymbol{\sigma}}(\mathbf{x}, t), \hat{\boldsymbol{\chi}}(\mathbf{x})) \, \mathrm{d}V \, \mathrm{d}t + \int_V \mu(\mathbf{x}) G(\hat{\boldsymbol{\chi}}(\mathbf{x})) \, \mathrm{d}V$$
$$+ \int_V \mathbf{v}(\mathbf{x}) \cdot \operatorname{div} \hat{\boldsymbol{\sigma}}^R(\mathbf{x}) \, \mathrm{d}V - \int_{S_t} \mathbf{v}(\mathbf{x}) \cdot \hat{\boldsymbol{\sigma}}^R(\mathbf{x}) \cdot \mathbf{n} \, \mathrm{d}S_t \tag{19}$$

where $\kappa > 0$ is some constant scalar introduced for dimensionality sake and $\dot{\lambda}(\mathbf{x}, t) \geqslant 0$, $\mu(\mathbf{x}) \geqslant 0$, $\mathbf{v}(\mathbf{x})$ are unknown Lagrangian multiplier functions, whose physical meaning will be clarified further on. Making use of the identity:

$$\int_V \mathbf{v}(\mathbf{x}) \cdot \operatorname{div} \hat{\boldsymbol{\sigma}}^R(\mathbf{x}) \, \mathrm{d}V = \int_S \mathbf{v}(\mathbf{x}) \cdot \hat{\boldsymbol{\sigma}}^R(\mathbf{x}) \cdot \mathbf{n} \, \mathrm{d}S - \int_V \hat{\boldsymbol{\sigma}}^R(\mathbf{x}) : \nabla^s \mathbf{v}(\mathbf{x}) \, \mathrm{d}V \tag{20}$$

(divergence theorem) where $\nabla^s(\cdot)$ is the symmetric part of the gradient operator, i.e. $\nabla^s \mathbf{v} = \frac{1}{2}(\operatorname{grad} \mathbf{v} + \operatorname{grad}^T \mathbf{v})$, substituting and reordering, the first variation of $\Upsilon = \Upsilon(s, \hat{\boldsymbol{\sigma}}^R, \hat{\boldsymbol{\chi}}, \dot{\lambda}, \mu, \mathbf{v})$ reads:

$$\delta\Upsilon = \left[-\kappa + \int_0^{\Delta T} \int_V \dot{\lambda} \frac{\partial f}{\partial \hat{\boldsymbol{\sigma}}} : \boldsymbol{\sigma}^E \, \mathrm{d}V \, \mathrm{d}t \right] \delta s$$
$$+ \int_V \left[\int_0^{\Delta T} \dot{\lambda} \frac{\partial f}{\partial \hat{\boldsymbol{\sigma}}} \, \mathrm{d}t - \nabla^s \mathbf{v} \right] \delta \hat{\boldsymbol{\sigma}}^R \, \mathrm{d}V + \int_{S_u} \mathbf{v} \cdot \delta \hat{\boldsymbol{\sigma}}^R \cdot \mathbf{n} \, \mathrm{d}S_u + \int_V \left[\int_0^{\Delta T} \dot{\lambda} \frac{\partial f}{\partial \hat{\boldsymbol{\chi}}} \, \mathrm{d}t + \mu \frac{\partial G}{\partial \hat{\boldsymbol{\chi}}} \right] \delta \hat{\boldsymbol{\chi}} \, \mathrm{d}V$$

$$+\int_0^{\Delta T}\int_V f(\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\chi}})\delta\dot{\lambda}\,\mathrm{d}V\,\mathrm{d}t+\int_V G(\hat{\boldsymbol{\chi}})\delta\mu\,\mathrm{d}V+\int_V \delta\mathbf{v}\cdot\operatorname{div}\hat{\boldsymbol{\sigma}}^R\,\mathrm{d}V-\int_{S_t}\delta\mathbf{v}\cdot\hat{\boldsymbol{\sigma}}^R\cdot\mathbf{n}\,\mathrm{d}S_t \tag{21}$$

where the arguments have been omitted for brevity. The Euler–Lagrange equations of problem (18a–f) turn out to be:

$$f(\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\chi}})\leqslant 0,\quad \dot{\lambda}\geqslant 0,\quad \dot{\lambda}f(\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\chi}})=0\quad \text{in } V\times[0,\Delta T] \tag{22a–c}$$

$$\hat{\boldsymbol{\sigma}}=s_a\boldsymbol{\sigma}^E+\hat{\boldsymbol{\sigma}}^R,\quad \dot{\boldsymbol{\varepsilon}}^p:=\dot{\lambda}\frac{\partial f}{\partial\hat{\boldsymbol{\sigma}}}\quad \text{in } V\times[0,\Delta T] \tag{22d,e}$$

$$G(\hat{\boldsymbol{\chi}})\leqslant 0,\quad \mu\geqslant 0,\quad \mu G(\hat{\boldsymbol{\chi}})=0\quad \text{in } V \tag{22f–h}$$

$$\Delta\varepsilon^p:=\int_0^{\Delta T}\dot{\boldsymbol{\varepsilon}}^p\,\mathrm{d}t\quad \text{in } V,\quad \Delta\varepsilon^p=\nabla^s\mathbf{v}\quad \text{in } V,\quad \mathbf{v}=\mathbf{0}\quad \text{on } S_u \tag{22i,k}$$

$$-\Delta\boldsymbol{\xi}:=\int_0^{\Delta T}\dot{\lambda}\frac{\partial f}{\partial\hat{\boldsymbol{\chi}}}\,\mathrm{d}t+\mu\frac{\partial G}{\partial\hat{\boldsymbol{\chi}}}\quad \text{in } V,\quad \Delta\boldsymbol{\xi}=\mathbf{0}\quad \text{in } V \tag{22l,m}$$

$$\operatorname{div}\hat{\boldsymbol{\sigma}}^R=\mathbf{0}\quad \text{in } V,\quad \hat{\boldsymbol{\sigma}}^R\cdot\mathbf{n}=\mathbf{0}\quad \text{on } S_t \tag{22n,o}$$

$$\int_0^{\Delta T}\int_V \boldsymbol{\sigma}^E:\dot{\boldsymbol{\varepsilon}}^p\,\mathrm{d}V\,\mathrm{d}t=\kappa>0 \tag{22p}$$

The physical meanings of the Lagrangian multipliers $\dot{\lambda}$ and μ arise as plastic and saturation multipliers, respectively. Equations (22a–c, f–h) are the plastic and saturation loading/unloading conditions while (22d,e) express the stress and plastic strain rates at $s = s_a$. $\Delta\boldsymbol{\varepsilon}^p$ and $\Delta\boldsymbol{\xi}$, given by eqns (22i) and (22l), have the meanings of cumulated plastic strains and kinematic internal variables over a complete cycle $[0, \Delta T]$, respectively. Equations (22j,k) express that the plastic strain increments $\Delta\boldsymbol{\varepsilon}^p$, over a complete cycle $[0, \Delta T]$, constitute a field compatible with the displacements $\mathbf{v}$ in V and zero displacements on S_u, while (22m) expresses the condition to be satisfied by the cumulated kinematic internal variables over a complete cycle. Equations (22n,o) are the already given conditions on the residual stresses $\hat{\boldsymbol{\sigma}}^R$. Finally, eqn (22p) expresses that the unamplified external actions do positive work through the plastic strain rates $\dot{\boldsymbol{\varepsilon}}^p$ in a complete cycle, Polizzotto et al. (1991).

The latter equations describe the shakedown limit state with its plastic deformation process and related impending inadaptation collapse, promoted by the shakedown limit loading $\mathbf{P}_a(t) \equiv s_a\bar{\mathbf{P}}(t)$. The pair $\Delta\boldsymbol{\varepsilon}^p$, $\Delta\boldsymbol{\xi}$ satisfying eqns (22j–k) and (22m), respectively, constitute a "plastic accumulation mechanism" (Polizzotto et al., 1991), or, alternatively, a "kinematically admissible plastic strain cycle" (Koiter, 1960).

Moreover, eqns (22a–p) are similar to the analogous equations for perfect plasticity presented in Panzeca and Polizzotto (1988) or the ones presented in Polizzotto et al. (1991) for elastic plastic materials with internal variables, as in fact in the present approach the only differences are the contribution to $\Delta\boldsymbol{\xi}$ coming from the adoption of a material model endowed with a saturation surface. The arguments aimed to assert the uniqueness features of the solution to (22a–p) are obviously the same of Polizzotto et al. (1991) for the analogous problem and are here omitted.

Nevertheless attention has to be focused on the physical understanding of the limit state described by eqns (22a–p).

As proved in Panzeca and Polizzotto (1988), in the case of perfectly plastic solids subjected to cyclic loads, the Euler–Lagrange eqns (22a–p) provide a solution which describes the gradient, with respect to the load multiplier, of the steady-state response of the solid body subjected to the cyclic loads at the shakedown limit state. The physical interpretation of such a (Euler–Lagrange) condition enables one to recognize the impending inadaptation collapse mechanism at the shakedown limit state (i.e. for $s = s_a$). This task is addressed in the next Section.

5. Impending inadaptation collapse mechanism at shakedown limit state

At the shakedown limit, i.e. at $s = s_a$, the kinematic variables have to satisfy conditions (22j,k,m). On taking into account the assumed expressions for the yield and the saturation functions, eqns (3) and (4), and remembering that $\boldsymbol{\chi} = (\mathbf{X}, Y)$, $\boldsymbol{\xi} = (\boldsymbol{\alpha}, \beta)$, we can write

$$\Delta\boldsymbol{\varepsilon}^p = \int_0^{\Delta T} \dot{\lambda} \frac{\partial f_0}{\partial \hat{\boldsymbol{\sigma}}} \mathrm{d}t \tag{23}$$

$$\Delta\boldsymbol{\alpha} = \int_0^{\Delta T} \dot{\lambda} \frac{\partial f_0}{\partial \hat{\mathbf{X}}} \mathrm{d}t - \mu \frac{\partial G_0}{\partial \hat{\mathbf{X}}} = \mathbf{0} \quad \text{in } V \tag{24}$$

$$\Delta\beta = \int_0^{\Delta T} \dot{\lambda} \frac{\partial f_1}{\partial \hat{Y}} \mathrm{d}t + \mu \frac{\partial G_1}{\partial \hat{Y}} = 0 \quad \text{in } V \tag{25}$$

where (23) expresses the cumulated plastic strains over a complete cycle $[0, \Delta T]$ (satisfying eqns (22j,k)) and (24), (25) express the cumulated kinematic internal variables over a complete cycle $[0, \Delta T]$ as a kinematic hardening, $\Delta\boldsymbol{\alpha}$, and an isotropic hardening, $\Delta\beta$, components, respectively. Equation (22m) has been accounted for in eqns (24) and (25). Making use of the identity $\partial f_0/\partial\hat{\boldsymbol{\sigma}} = -\partial f_0/\partial\hat{\mathbf{X}}$ and remarking that $f_1(\hat{Y})$ is time-independent, by substituting (23) in (24) and rearranging we get:

$$\Delta\boldsymbol{\varepsilon}^p = -\mu \frac{\partial G_0}{\partial \hat{\mathbf{X}}} \quad \text{in } V \tag{26}$$

$$\Delta\lambda \frac{\partial f_1}{\partial \hat{Y}} = -\mu \frac{\partial G_1}{\partial \hat{Y}} \quad \text{in } V \tag{27}$$

with $\Delta\lambda = \int_0^{\Delta T} \dot{\lambda}\, \mathrm{d}t > 0$ [note that $\dot{\lambda} = 0$ everywhere in $V \times [0, \Delta T]$ would be in contrast with (22p)] and $\mu \geqslant 0$. In what follows we discuss the impending inadaptation collapse mechanism at $s = s_a$ on the basis of eqns (26) and (27). The straightforward consequences on the remaining eqns (22a–h,n–p) are omitted for brevity.

In the assumed hypothesis of a positive-isotropic/kinematic hardening material endowed with a saturation surface of general shape as the one depicted in Fig. 1(a), two cases can be distinguished according to whether the material exhibits or not isotropic hardening.

Case 1: Nonzero isotropic hardening.

In this case it is $\partial f_1/\partial \hat{Y} > 0 \,\forall \hat{Y} \geqslant 0$ *in* V and thus, from eqn (27) we get:

$$\mu > 0 \quad \text{in } V, \quad \frac{\partial G_1}{\partial \hat{Y}} < 0 \quad \text{in } V \tag{28}$$

The first of eqn (28) implies $G(\hat{\mathbf{X}}, \hat{Y}) = 0$, i.e. $(\hat{\mathbf{X}}, \hat{Y})$ is a saturated state; whereas the second eqn (28) implies $\hat{Y} > \tilde{Y}$, $\tilde{Y}$ being the value at which $\partial G_1/\partial \hat{Y} = 0$ (refer also to Fig. 1(a)). Also, from (26) we have: (i) for all the saturated states $(\hat{\mathbf{X}}, \hat{Y})$ with $\hat{Y} > \tilde{Y}$ at which $\partial G_0/\partial \hat{\mathbf{X}} \neq 0$ in V, it is $\Delta \boldsymbol{\varepsilon}^p \neq \mathbf{0}$ in V, hence *ratchetting* occurs, and (ii) for all the saturated states $(\hat{\mathbf{X}}, \hat{Y})$ with $\hat{Y} > \tilde{Y}$ at which $\partial G_0/\partial \hat{\mathbf{X}} = 0$ in V, it is $\Delta \boldsymbol{\varepsilon}^p = \mathbf{0}$ in V, hence *alternating plasticity* occurs (e.g. points $\hat{\mathbf{X}}$, $\hat{Y}$ over the plateau $Y = Y^0$ in Fig. 1(a)). Obviously, at the edge points on the plateau $Y = Y^0$ both mechanisms can take place.

Case 2: Zero isotropic hardening.

In this case it is $\partial f_1/\partial \hat{Y} = 0 \,\forall \hat{Y} \geqslant 0$ in V, and thus from eqn (27) we get either:

$$\mu = 0 \quad \text{in } V, \quad \frac{\partial G_1}{\partial \hat{Y}} \neq 0 \quad \text{in } V \tag{29a}$$

or

$$\mu \neq 0 \quad \text{in } V, \quad \frac{\partial G_1}{\partial \hat{Y}} = 0 \quad \text{in } V \tag{29b}$$

or even

$$\mu = \frac{\partial G_1}{\partial \hat{Y}} = 0 \quad \text{in } V \tag{29c}$$

If eqn (29a) is valid we have $G < 0$ in V, i.e. the material hardening state is non-saturated and from (26) we get $\Delta \boldsymbol{\varepsilon}^p = \mathbf{0}$ in V, hence *alternating plasticity* occurs for all states $(\hat{\mathbf{X}}, \hat{Y})$ inside $G = 0$. If (29b) is valid we have: $\hat{Y} \equiv \tilde{Y}$, $G = 0$ and being $\partial G_0/\partial \hat{\mathbf{X}}\,|_{\tilde{Y}} \neq 0$ from (26) we get $\Delta \boldsymbol{\varepsilon}^p \neq \mathbf{0}$ in V, hence *ratchetting* occurs for all the saturated states at $\hat{Y} = \tilde{Y}$. Finally if (29c) is valid, *alternating plasticity* occurs once again for all the non-saturated states inside $G = 0$ at $\hat{Y} = \tilde{Y}$. The types of impending inadaptation mechanism modes allowed at $s = s_a$ are so fully described grounding on eqns (26) and (27).

Summarizing, one can state that, with the present material model, the impending inadaptation collapse mode is reverse plasticity if the limit hardening state $\hat{\mathbf{X}}$, $\hat{Y}$ is nonsaturated, and ratchetting if this limit state is saturated, except over the plateau (if any) $Y = Y^0$, where reverse plasticity is also allowed.

6. Two-degrees-of-freedom bar structure

In order to show the main features of the cyclic response of a structure made of an elastic–plastic material obeying the discussed constitutive model, a numerical example is hereafter presented. A

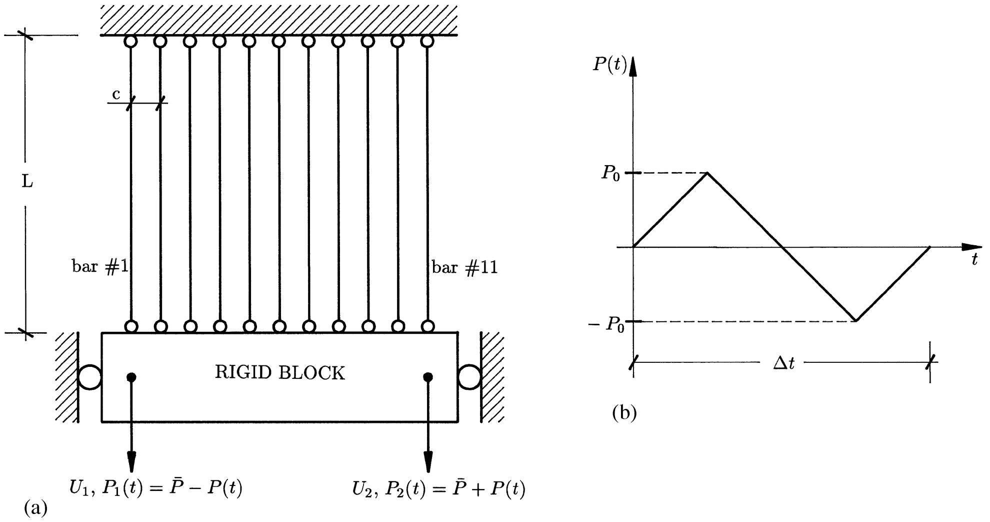

Fig. 5. Two-degrees-of-freedom bar structure subjected to cyclic load: (a) geometry and loading scheme; and (b) load history.

simple structure consisting of 11 parallel pinned bars, all of equal length $L = 20$ cm, fixed at the upper end and joined to a rigid block at the lower end, Fig. 5(a), is considered. The bars have equal cross section of area $A = 1$ cm^2 and are located at constant intervals $c = 2$ cm. The structure possesses two degrees of freedom which are assumed as the vertical displacements U_1 and U_2 at the left and right ends of the rigid block, respectively. Consistently with the degrees of freedom two loads, $P_1(t) = \bar{P} - P(t)$ and $P_2(t) = \bar{P} + P(t)$, are applied. $\bar{P}$ is a constant-value load and $P(t)$ varies in time between the closed interval $[-P_0, P_0]$ in the shape given in Fig. 5(b).

The bars are all made of an elastic–plastic material endowed with saturation surface obeying the constitutive laws presented in Section 2 particularized for the uniaxial case. Namely, it has been assumed $\Psi = B(\alpha^2 + a\beta^2)/2$, $f = |\sigma - X| - \sigma_y$, $G = |X| - G_1(Y)$; with $G_1(Y) = c_0\sigma_y + Y(1 - c_1 Y/Y_L)$ and constants values as: $c_0 = 0.5$, $c_1 = 0.4$, $Y_L = 20$; also, $\sigma_y = 20$ kN/cm^2, $E = 21{,}000$ kN/cm^2, $B = E/3$, $a = 2/3$. The stress–strain curve for monotonically increasing strain is shown as line OAB in Fig. 6, where the perfect plasticity stage is reached at $\sigma = 42$ kN/cm^2 (at which a saturated state is attained). In the same figure the stress response to a periodic strain history, $-0.7\% \leqslant \varepsilon \leqslant 0.7\%$, is also reported. Due to the relatively large strain amplitude the saturated condition is periodically attained whether in tension or in compression.

Despite its simplicity the examined structure is able to reproduce all the expected inadaptation modes including the ratchetting behaviour which is here obtainable for the presence of a bound on the hardening capacity of the material, as shown in the following. A fully implicit step-by-step analysis of the structure of Fig. 5(a) has been carried out for eight loading cycles and for three different loading conditions; namely:

—load condition (A): $\bar{P} = 125$ kN, $P_0 = 40$ kN;
—load condition (B): $\bar{P} = 75$ kN, $P_0 = 75$ kN;
—load condition (C): $\bar{P} = 125$ kN, $P_0 = 75$ kN.

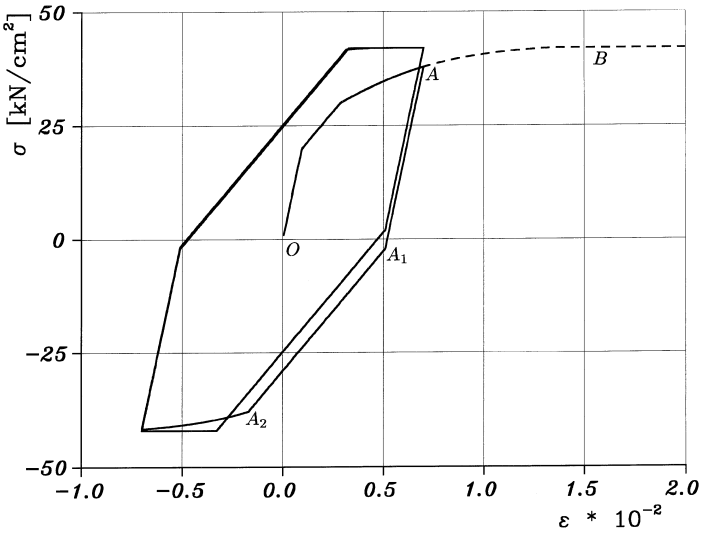

Fig. 6. Stress–strain curve of a uniaxial specimen loaded by a monotonically increasing total strain (curve OAB) and by a cyclic total strain (curve $OAA_1A_2\ldots$).

The results obtained are reported in Figs 7–9. In each figure the responses concerning the applied loading conditions are represented as: small-dashed line [condition (A)], dashed line [condition (B)] and solid line [condition (C)], respectively.

Condition (A) is characterized by a purely elastic steady-state response, that is shakedown. Figure 7 shows the displacements–cycles curves which, starting from the third cycle exhibit a linear elastic cyclic behaviour. The same behaviour is displayed in Fig. 8 where the load–displacement curves are reported. Figure 9 shows the plastic strain cyclic evolution at the extreme bars, bar #1 and #11, and again the shakedown condition transpires by the circumstance that plastic strain ceases starting from the third cycle.

Condition (B) gives rise to alternating plasticity response mode. As in fact the displacement–cycles curves (Fig. 7) exhibit a nonlinear alternating cyclic behaviour. The load–displacement curves (Fig. 8) show the elastic plastic steady response in the shape of a structural periodic hysteretic cycle. The plastic strain evolution at bar #1 and #11, Fig. 9, is in fact characterized by an alternating plastic strain production with a zero net plastic strain accumulation in the steady-cycle.

Condition (C) produces a ratchetting response mode; namely, the displacement–cycles curves, Fig. 7, exhibit a nonlinear cyclic behaviour with a net displacement cyclic growth. The same structural behaviour can be detected by inspection of the load–displacement curves in Fig. 8 and of the plastic strain evolution curves at bars #1 and #11 reported in Fig. 9.

Finally, on taking into account that the loading conditions considered in Fig. 5 for the examined structure are fully equivalent to a constant load, $2\bar{P}$, and a variable couple, $P(t)L\ t\in[0,\Delta t]$, both applied at the mid point of the rigid block, the shakedown load multiplier s_a has been computed grounding on problem (18a–f). To this aim, let $\bar{\sigma}_i^E$ and $\tilde{\sigma}_i^E(t)\ t\in[0,\Delta t]$ be the elastic stress responses to the constant load and the variable couple in the ith bar, respectively, and let $\sigma_i^E(t) := \bar{\sigma}_i^E + s\tilde{\sigma}_i^E(t)\ t\in[0,\Delta t]$ be the elastic response to the constant load $2\bar{P}$ and to the amplified variable couple, $sP(t)L$. Problem (18a–f) can thus, be particularized as follows:

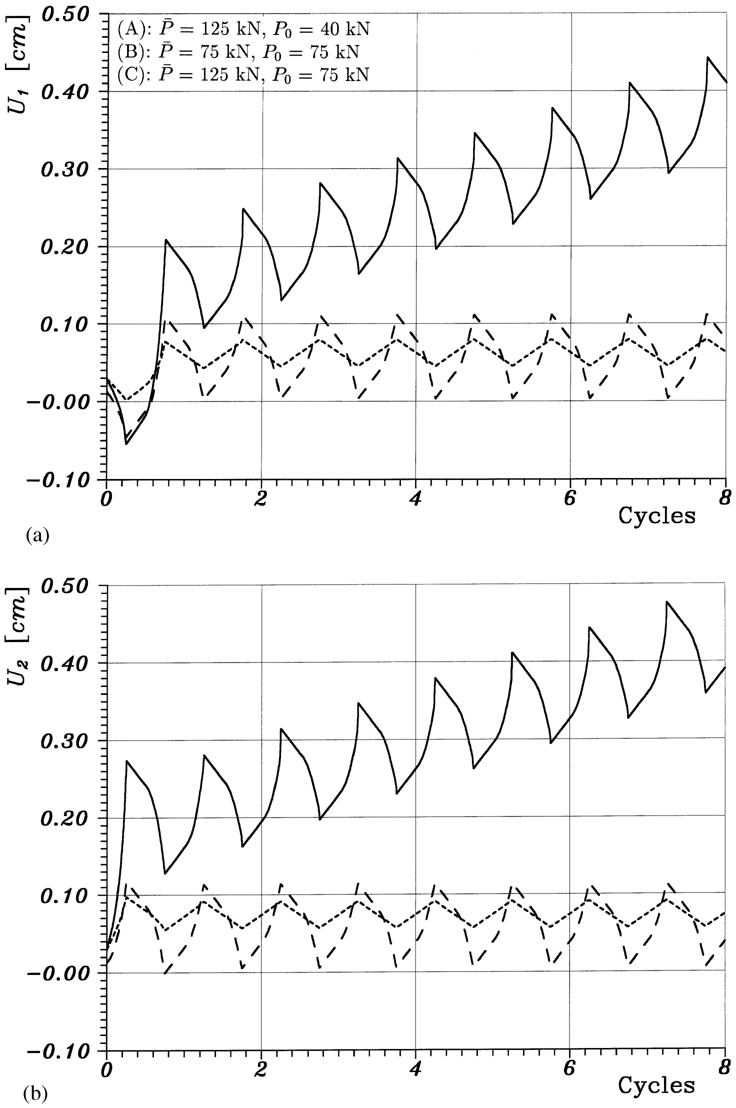

Fig. 7. Response of the structure of Fig. 5 to the loading conditions (A) small-dashed line, (B) dashed line and (C) solid line: (a) rigid block displacement U_1 history; and (b) rigid block displacement U_2 history.

$$s_a = \max_{(s,\rho_i,X_i)} s \quad \text{subject to:} \sigma_i^{E+} + \rho_i - X_i \leqslant \sigma_y; \quad -(\sigma_i^{E-} + \rho_i - X_i) \leqslant \sigma_y;$$

$$|X_i| \leqslant G_1(Y_L); \quad \sum_{i=1}^{n_b} A_i \rho_i = 0; \quad \text{for } i = 1, \ldots, n_b \tag{30}$$

where n_b = number of bars, ρ_i = self-stress in the ith bar while σ_i^{E+} and σ_i^{E-} denote the maximum and the minimum elastic stress values attained in the ith bar in $[0, \Delta t]$, respectively. It is worth

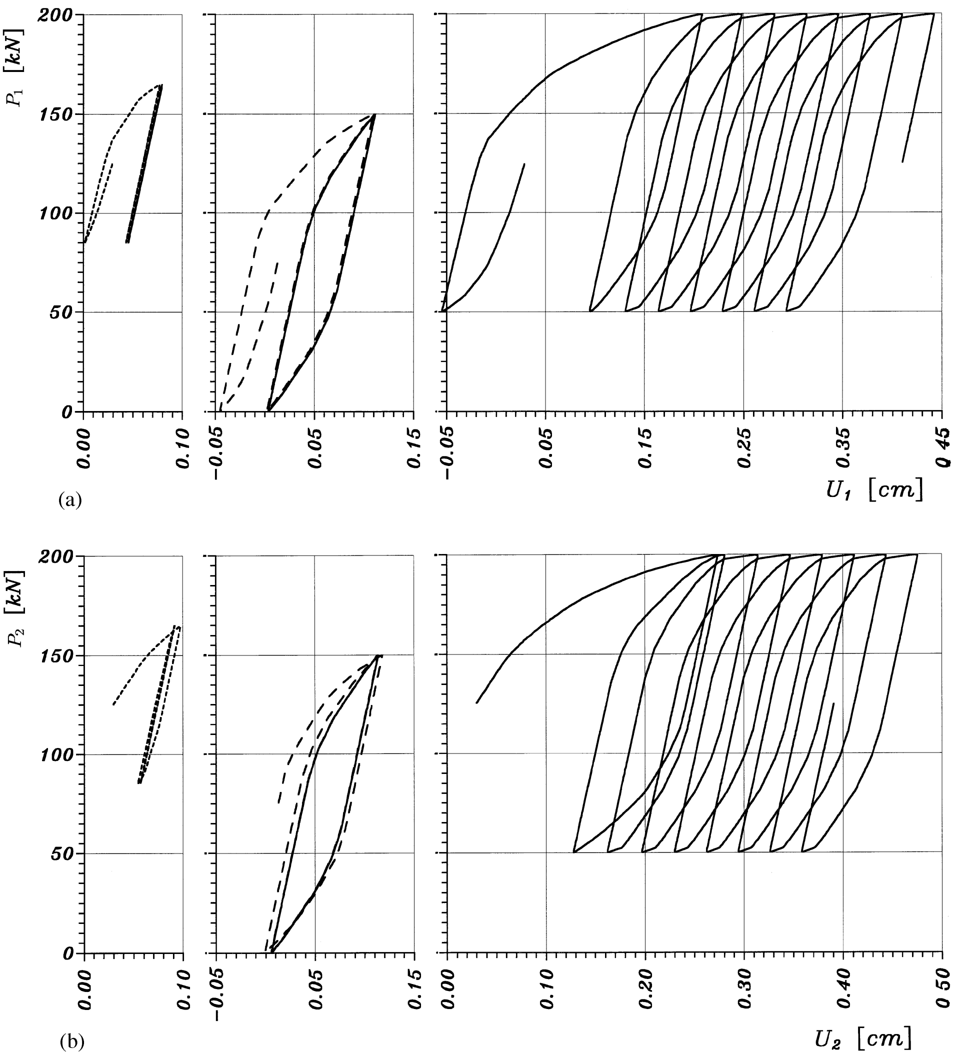

Fig. 8. Response of the structure of Fig. 5 to the loading conditions (A) small-dashed line, (B) dashed line and (C) solid line: (a) $P_1 - U_1$ curve; and (b) $P_2 - U_2$ curve.

noting that the saturation condition $|X_i| \leqslant G_1(Y_i)$ arising from problem (18a–f) for the examined case can be enforced, for the assumed material model, as $|X_i| \leqslant G_1(Y_L)$ so obtaining a simpler linear programming problem. Figure 10 shows, in a dimensionless form, the interaction (or generalized Bree) diagram of the given structure as the amplitude of the variable couple P_0L/M^E vs the constant load $2\bar{P}/P_U$, where $M^E = 880$ kNcm is the elastic limit couple and $P_U = 462$ kN in the plastic limit constant load. The shakedown limit load curve, reported as solid line, exhibits a constant branch pertaining to impending reverse plasticity inadaptation mode and a linear branch pertaining to impending ratchetting inadaptation mode. It is worth noting that this latter branch can exist only if a material endowed with saturation surface is considered, as in fact, for a material

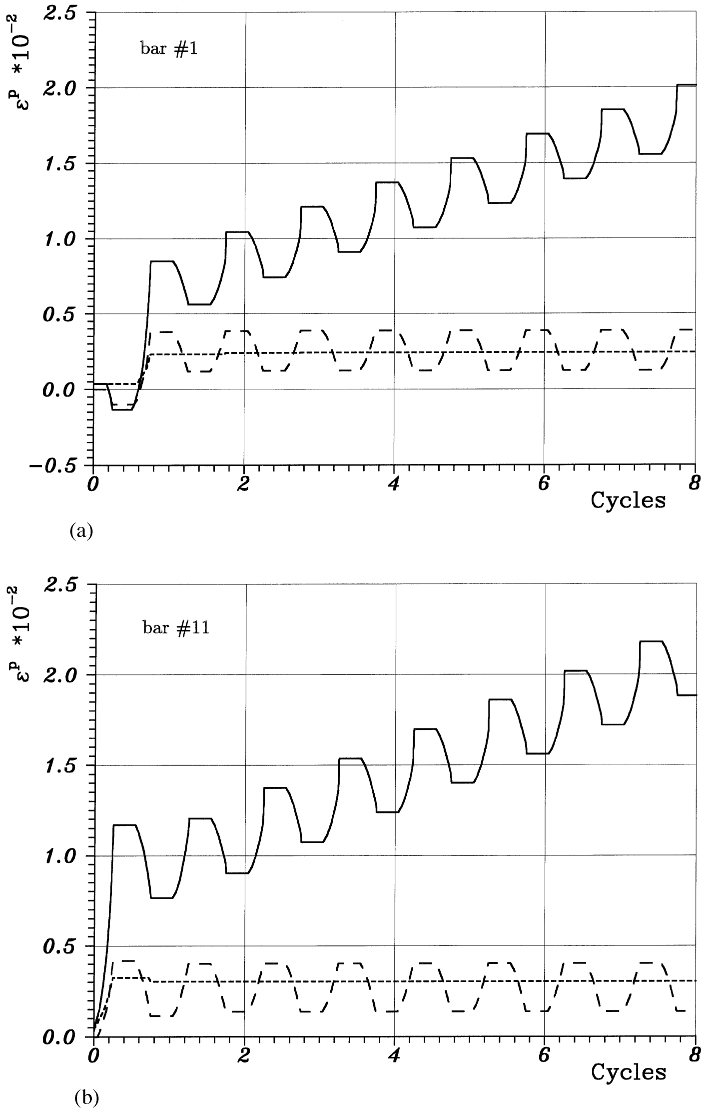

Fig. 9. Response of the structure of Fig. 5 to the loading conditions (A) small-dashed line, (B) dashed line and (C) solid line: (a) plastic strain evolution at bar #1; and (b) plastic strain evolution at bar #11.

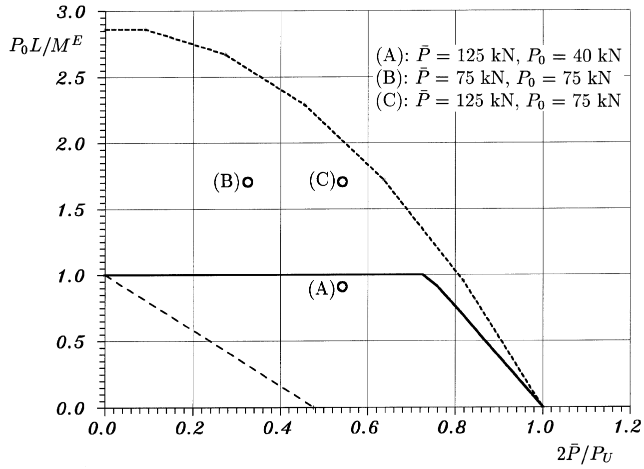

Fig. 10. Interaction diagram for the bar structure of Fig. 5: shakedown limit load curve (solid line), plastic collapse limit load curve (small-dashed line) elastic limit load curve (dashed line) and analyzed load conditions.

without saturation limit condition, only the constant unbounded branch would exist, the latter giving rise just to alternating plasticity impending inadaptation mode and this at any fixed constant load value. The plastic collapse limit load curve and the elastic limit load curve are also reported as small-dashed line and dashed line, respectively, as well as the load conditions (A), (B) and (C) previously analyzed.

7. Concluding remarks

The main findings of the present study can be synthesized as follows:

(1) A recently developed elastic–plastic internal variable constitutive model with hardening saturation surface has been implemented into the shakedown theory. The adoption of a saturation surface in the internal variable space results in an effective bound on the material hardening capacity, so that a perfectly-plastic behaviour can be reached in a saturated limit state.
(2) A statical Melan-type shakedown theorem has been formulated and proved for the considered material model.
(3) The impending inadaptation collapse mechanism at shakedown limit state has been analyzed starting from the optimality conditions relative to the determination of the shakedown load multiplier, the latter performed in a statical fashion.
(4) The present approach allows to fully describe the possible inadaptation collapse mechanisms, i.e. alternating plasticity and ratchetting, overtaking all the drawbacks related to unlimited isotropic/kinematic hardening models.

It is worth noting that similar results are obtainable with approaches based on the assumption of a bounding/limit surface in the stress space—which turns out to be a bound on the hardening capacity of the material—but these approaches, to the author's knowledge, are available for materials obeying very simple constitutive models (Von Mises-type models). On the contrary the proposed approach—which bounds the hardening capacity of the material in the internal variable space—is quite general and appears to be suitable for a rather large class of constitutive models for elastic–plastic hardening materials.

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